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A LINEAR THEORY OF THIN ELASTIC SHELLS, BASED ON CONSERVATION OF A NON-NORMAL STRAIGHT LINE

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Abstract—A complete first-approximation linear shell theory is presented, based on the kinematical assumption that through every point of the reference surface a given material straight line, not necessarily normal to it, remains straight after deformation. The equations are formulated in tensor notation.

NOTATION

a, b, c, d, e	indices for cartesian components
A	area
$A^{\alpha}_{\beta\gamma}$	surface tensor related to deviation from normal
$B^{\cdot\beta}_{\alpha}$	surface tensor (generalized curvature)
C	curve on the reference surface
C_{α}	surface vector related to deviation from normal
$D_{\alpha\beta}$	surface tensor (generalized curvature)
e^{a}_{i}	the ath cartesian component of the ith base vector of frame ξ^1, ξ^2, ξ^3
$\tilde{E}_{ij}$	three-dimensional strain tensor
$E_{\alpha\beta}$	surface (membrane) strain tensor
$E_{\alpha 3} = E_{3\alpha}$	surface (shear) strain vector
E_{33}	surface (transverse) strain invariant
EVW	external virtual work
f^a, g^a	functions of ξ^1, ξ^2 describing geometry of shell
g	determinant of $G_{\alpha\beta}$
$\tilde{G}$	determinant of G_{ij}
$\tilde{G}_{ij}$	three-dimensional metric tensor
$G_{\alpha\beta}$	surface metric tensor
$G_{\alpha 3} = G_{3\alpha}$	surface metric vector
G_{33}	surface metric invariant
$H^{\alpha\beta}$	tensor defined in equation (17)
i, j, k, l, m	indices for components in frame ξ^1, ξ^2, ξ^3
$\{ij, k\}$	three-dimensional Christoffel symbols of the first kind of frame ξ^1, ξ^2, ξ^3 at $\xi^3 = 0$
IVW	internal virtual work
$K_{\alpha\beta}$	surface (bending) strain tensor
m^a	ath cartesian component of the external moment/unit area of the reference surface
m_{α}	components of the external moment/unit area of the reference surface in frame ξ^1, ξ^2, ξ^3
M^a	ath cartesian component of the moment resultant/unit length of a curve on the reference surface
$M^{\alpha\beta}$	two-dimensional moment tensor
$N^{\alpha\beta}$	two-dimensional (membrane) force tensor
$\bar{N}^{\alpha\beta}$	modified two-dimensional force tensor [equation (60)]

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p^a	ath cartesian component of the external force/unit area of the reference surface
p_i	components of the external force/unit area of the reference surface in frame ξ^1, ξ^2, ξ^3
Q^α	two-dimensional (shear) force vector
$\bar{Q}^\alpha$	modified two-dimensional force vector [equation (61)]
r^a, u^a	functions of ξ^1, ξ^2 describing displacement of shell
r_i, u_i	kinematic unknowns
R^a	ath cartesian component of the force resultant/unit length of a curve on the reference surface
$\tilde{T}^{ijkm}$	three-dimensional tensor of elastic coefficients appearing in three-dimensional constitutive equations
x^1, x^2, x^3	cartesian coordinates
$\{\alpha\beta, \gamma\}$	two-dimensional Christoffel symbols of the first kind corresponding to metric tensor $G_{\alpha\beta}$
δ^i_j	Kronecker delta
ε_{ijk}	alternating three-dimensional tensor in frame ξ^1, ξ^2, ξ^3
$\varepsilon_{\alpha\beta}$	alternating surface tensor
λ_α	unit outward normal vector to C in reference surface
ξ^1, ξ^2, ξ^3	shell coordinates
$\Pi^\alpha_{\beta\gamma}$	two-dimensional Christoffel symbols of the second kind corresponding to metric tensor $G_{\alpha\beta}$ [equation (16)]
$\tilde{\sigma}^{ij}$	three-dimensional stress tensor
Γ^i_{jk}	three-dimensional Christoffel symbols of the second kind of frame ξ^1, ξ^2, ξ^3 at $\xi^3 = 0$

INTRODUCTION

THE classical linear thin-shell theory—formulated originally by Love [6] and subsequently improved by Koiter [5] and by Budiansky and Sanders [1]—is based on the following assumptions: (1) the shell thickness at any point of the reference surface is small compared with the minimum radius of curvature of the latter at that point; (2) normals to the pre-deformation reference surface are rigid and remain normal to its post-deformation counterpart (Kirchhoff–Love hypothesis).

While the first assumption defines the basis of the theory, justifying replacement of the triaxial state of stress by a biaxial state of forces and moments, the second is a restriction specifying some preferential character for the normals to the reference surface before and after deformation. The physical meaning of that restriction is that (as in the classical beam theory) shear deformations are not taken into account, which implies that the theory cannot cover: (1) such structures as sandwich shells; (2) dynamic behaviour, transverse waves being ruled out. In those circumstances, a consistent theory— which may be regarded as Timoshenko's beam theory [9] extended to shells—was formulated [8], based on substituting a Bernoulli-type hypothesis for the Kirchhoff–Love one, namely: normals to the pre-deformation reference surface are rigid but do not necessarily retain their normality after deformation. That alternative theory, however, is still inapplicable to shells cut out from layered materials, in view of their naturally preferred directions which are not necessarily normal to the reference surface, nor can it serve as basis for a geometrically nonlinear theory admissive of large shear strains, since the deformed structure no longer qualifies as a shell under Bernoulli's hypothesis.

The linear theory presented below is based on a less restrictive hypothesis, namely that through every point of the reference surface a given material straight line, not necessarily normal to it, remains straight after deformation. The theory is formulated in tensor notation, so that its equations are valid in any coordinate system on the reference surface. Like the classical shell theory, it recognizes the triaxial state of stress, integrated over the thickness, and refrains accordingly from treating the shell as a generalized continuum [2–4].

MATHEMATICS

(a) *Basic definitions*

Using the straight line in the preferred direction as the third axis of coordinates to those of the reference surface, a point on the latter being represented by:

$$x^a = f^a(\xi^1, \xi^2) \tag{1}$$

a point within the shell is given by:

$$\tilde{x}^a = f^a(\xi^1, \xi^2) + \xi^3 g^a(\xi^1, \xi^2) \tag{2}$$

where f^a, g^a are functions of ξ^1, ξ^2 alone.

We are concerned with coordinate transformations of the form:

$$\begin{cases} \bar{\xi}^1 = \bar{\xi}^1(\xi^1, \xi^2) \\ \bar{\xi}^2 = \bar{\xi}^2(\xi^1, \xi^2) \\ \bar{\xi}^3 = \xi^3 \end{cases} \tag{3}$$

for which the law of transformation of the contravariant components F^i of a vector are:

$$\begin{aligned} \bar{F}^\alpha &= F^\beta \frac{\partial \bar{\xi}^\alpha}{\partial \xi^\beta} \\ \bar{F}^3 &= F^3. \end{aligned} \tag{4}$$

This is the same as saying that a three-dimensional vector F^i is composed of a two-dimensional (or surface) vector F^α and an invariant F^3 with respect to transformations of type (3); similarly, a tensor T^{ij} can be resolved into a surface tensor $T^{\alpha\beta}$, two surface vectors $T^{\alpha 3}$ and $T^{3\alpha}$ and a surface invariant T^{33}.

The covariant base vectors, at every point of the shell, are given by:

$$\tilde{e}^a_\alpha = \frac{\partial \tilde{x}^a}{\partial \xi^\alpha} = \frac{\partial f^a}{\partial \xi^\alpha} + \xi^3 \frac{\partial g^a}{\partial \xi^\alpha} = e^a_\alpha + \xi^3 \frac{\partial g^a}{\partial \xi^\alpha} \tag{5}$$

$$\tilde{e}^a_3 = \frac{\partial \tilde{x}^a}{\partial \xi^3} = g^a = e^a_3$$

in which equation (2) was used.

The metric tensor is given by:

$$\tilde{G}_{ij} = \tilde{e}^a_i \tilde{e}^a_j \tag{7}$$

and by the above considerations can be resolved into a surface metric tensor:

$$\tilde{G}_{\alpha\beta} = \tilde{e}^a_\alpha \tilde{e}^a_\beta \tag{8}$$

two (equal) surface vectors:

$$\tilde{G}_{\alpha 3} = \tilde{G}_{3\alpha} = \tilde{e}^a_\alpha \tilde{e}^a_3 \tag{9}$$

and a surface invariant:

$$\tilde{G}_{33} = \tilde{e}^a_3 \tilde{e}^a_3. \tag{10}$$

At every point, the contravariant three-dimensional base vectors are defined by:

$$\tilde{e}^{ai} = \tilde{G}^{ij}\tilde{e}^{a}_{j} \tag{11}$$

in which $\tilde{G}^{ij}$ is obtained from:

$$\tilde{G}^{ik}\tilde{G}_{km} = \delta^{i}_{m}. \tag{12}$$

(b) *Christoffel symbols*

Two different metrics have been defined. The first, $\tilde{G}^{ij}$, corresponds to the triaxial shell coordinates and is given by equation (7). For these metrics the first- and second-kind Christoffel symbols at $\xi^3 = 0$ are defined, respectively, by:

$$\{ij, k\} = \frac{1}{2}\left[\frac{\partial G_{ik}}{\partial \xi^{j}} + \frac{\partial G_{jk}}{\partial \xi^{i}} - \frac{\partial G_{ij}}{\partial \xi^{k}}\right] \tag{13}$$

and:

$$\Gamma^{i}_{jk} = G^{il}\{jk, l\}. \tag{14}$$

For the metrics $\tilde{G}_{\alpha\beta}$, defined by equation (8) and corresponding to a two-dimensional manifold, the first-kind Christoffel symbols at $\xi^3 = 0$ are given by:

$$\{\alpha\beta, \gamma\} = \frac{1}{2}\left[\frac{\partial G_{\alpha\gamma}}{\partial \xi^{\beta}} + \frac{\partial G_{\beta\gamma}}{\partial \xi^{\alpha}} - \frac{\partial G_{\alpha\beta}}{\partial \xi^{\gamma}}\right]. \tag{15}$$

Note that since $G_{\alpha\beta}$ is a submatrix of G_{ij}, $\{\alpha\beta, \gamma\}$ equals $\{ij, k\}$ for $i = \alpha$, $j = \beta$ and $k = \gamma$. On the other hand, the second-kind Christoffel symbols for the surface metrics at $\xi^3 = 0$ are given by:

$$\Pi^{\alpha}_{\beta\gamma} = H^{\alpha\varphi}\{\beta\gamma, \varphi\} \tag{16}$$

where $H^{\alpha\varphi}$ is obtained from:

$$H^{\alpha\varphi}G_{\varphi\omega} = \delta^{\alpha}_{\omega}. \tag{17}$$

Comparing equations (16) and (17) with equations (12) and (14), it can be concluded that in general:

$$\Pi^{\alpha}_{\beta\gamma} \neq \Gamma^{\alpha}_{\beta\gamma}. \tag{18}$$

The second-kind Christoffel symbols are related to the derivatives of the base vectors by:

$$\frac{\partial e^{a}_{i}}{\partial \xi^{j}} = \Gamma^{k}_{ij}e^{a}_{k}. \tag{19}$$

As is known Γ^{i}_{jk} is not a tensor. Still the restricted symbols $\Gamma^{\beta}_{3\alpha}$, $\Gamma^{3}_{3\alpha}$ and $\Gamma^{3}_{\alpha\beta}$ represent surface tensors and vectors. This follows immediately by applying equations (3) to the law of transformation of Christoffel symbols (see e.g. [7]). Similarly, it can be proved that the difference:

$$A^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Pi^{\alpha}_{\beta\gamma} \tag{20}$$

is a surface tensor.

Denoting:

$$B_{\alpha}^{\cdot\beta} = \Gamma_{3\alpha}^{\beta} \tag{21}$$

$$C_{\alpha} = \Gamma_{3\alpha}^{3} \tag{22}$$

$$D_{\alpha\beta} = \Gamma_{\alpha\beta}^{3} \tag{23}$$

the derivatives of the base vectors at $\xi^3 = 0$ are given by:

$$\frac{\partial e_{\alpha}^{a}}{\partial \xi^{\beta}} = \Gamma_{\alpha\beta}^{\gamma} e_{\gamma}^{a} + D_{\alpha\beta} e_{3}^{a} \tag{24}$$

$$\frac{\partial e_{3}^{a}}{\partial \xi^{\alpha}} = \frac{\partial e_{\alpha}^{a}}{\partial \xi^{3}} = B_{\alpha}^{\cdot\beta} e_{\beta}^{a} + C_{\alpha} e_{3}^{a} \tag{25}$$

$$\frac{\partial e_{3}^{a}}{\partial \xi^{3}} = 0. \tag{26}$$

It should be noted that if e_3^a is identified everywhere with the unit normal to the reference surface, the tensors $A_{\beta\gamma}^{\alpha}$ and C_{α} vanish identically, while $B_{\alpha}^{\cdot\beta}$ and $D_{\alpha\beta}$ reduce to the tensor of curvature of the surface. Equations (24) and (25) can therefore be viewed as a generalization of Gauss' and Weingarten's formulae, respectively.

(c) *Theorem of surface divergence*

The surface divergence theorem (see e.g. [7]) states:

$$\oint_C F^{\alpha}\lambda_{\alpha}\, \mathrm{d}s = \iint_A F^{\alpha}_{;\alpha}\, \mathrm{d}A \tag{27}$$

where F^{α} is any surface vector; C is a closed curve on the surface; A is the area enclosed by it; a semicolon denotes the covariant derivative with respect to the surface metrics (i.e. with the use of $\Pi_{\beta\gamma}^{\alpha}$); and λ_{α}—the unit outward normal vector to C in the surface—is given by:

$$\lambda_{\alpha} = \sqrt{\left(\frac{g}{G}\right)}\varepsilon_{\alpha\beta 3}\frac{\mathrm{d}\xi^{\beta}}{\mathrm{d}s} = \varepsilon_{\alpha\beta}\frac{\mathrm{d}\xi^{\beta}}{\mathrm{d}s} \tag{28}$$

with g = determinant of $G_{\alpha\beta}$; G = determinant of G_{ij}; ε_{ijk} = alternating three-dimensional tensor; $\varepsilon_{\alpha\beta}$ = alternating surface tensor. (Note that $\sqrt{(g/G)} = \sqrt{(G^{33})}$ is a surface invariant).

TWO DIMENSIONAL FORCES AND MOMENTS

(a) *Two dimensional forces*

An element of the shell at point ξ_0^1, ξ_0^2 of the reference surface, cut through by a plane containing the straight line

$$\tilde{x}^a = f^a(\xi_0^1, \xi_0^2) + \xi^3 g^a(\xi_0^1, \xi_0^2) \tag{29}$$

is shown in Fig. 1.

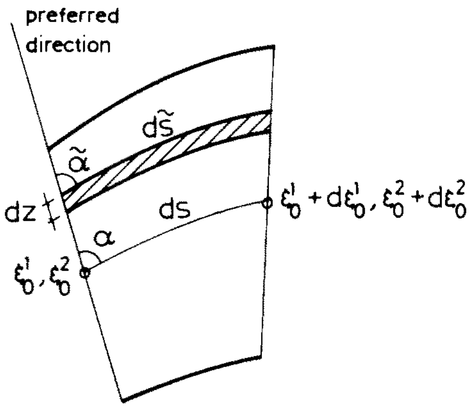

FIG. 1. Shell element.

Defining a coordinate z which measures length along ξ^3, the area shaded in Fig. 1 is given by:

$$\text{area} = dz\, d\tilde{s} \sin\tilde{\alpha}. \tag{30}$$

The Cartesian components of the resultant of the internal forces acting on the elementary section can be calculated as:

$$R^a\, ds = \int_z \tilde{\sigma}^{ba}\tilde{n}_b\, dz\, d\tilde{s} \sin\tilde{\alpha} \tag{31}$$

where $\tilde{\sigma}^{ba}$ = Cartesian components of three-dimensional stress tensor;
$\tilde{n}_b$ = Cartesian components of a unit vector normal to plane of section.

Taking into account the following equality:

$$\varepsilon_{bcd}e_3^d \frac{d\xi^\alpha}{d\tilde{s}} \frac{d\tilde{x}^c}{d\xi^\alpha} d\xi^3 = \tilde{n}_b \sin\tilde{\alpha}\, dz \tag{32}$$

and expressing the Cartesian components of the three-dimensional stress tensor in terms of the local coordinate components, the following expression is obtained from equation (31):

$$R^a = N^{\beta\rho}\lambda_\beta e_\rho^a + Q^\beta\lambda_\beta e_3^a \tag{33}$$

where

$$N^{\beta\rho} = \int_{\xi^3} \sqrt{\left(\frac{\tilde{G}}{g}\right)} \tilde{\sigma}^{\beta\varphi}[\delta_\varphi^\rho + \xi^3 B_{\varphi}^{\cdot\rho}]\, d\xi^3 \tag{34}$$

and

$$Q^\beta = \int_{\xi^3} \sqrt{\left(\frac{\tilde{G}}{g}\right)} [\tilde{\sigma}^{\beta 3} + \xi^3\tilde{\sigma}^{\beta\varphi}C_\varphi]\, d\xi^3. \tag{35}$$

Equations (34) and (35) define the biaxial forces.

(b) *Two dimensional moments*

The moment of the internal forces acting on the elementary section shown in Fig. 1 with respect to point (ξ_0^1, ξ_0^2) is given in Cartesian components by:

$$M^a\, ds = \int_z \varepsilon_{.cd}^a \xi^3 e_3^c \tilde{\sigma}^{bd}\tilde{n}_b\, dz\, d\tilde{s} \sin\tilde{\alpha}. \tag{36}$$

Using equation (32) and expressing the three-dimensional stress tensor in local coordinate components, the following expression is obtained from equation (36):

$$M^{a} = M^{\beta\rho}\varepsilon_{\rho\omega 3}\lambda_{\beta}\,\mathrm{e}^{a\omega} \tag{37}$$

with:

$$M^{\beta\rho} = \int_{\xi^3} \sqrt{\left(\frac{\tilde{G}}{g}\right)}\tilde{\sigma}^{\beta\varphi}(\delta^{\rho}_{\varphi}+\xi^3 B^{\cdot\rho}_{\varphi\cdot})\xi^3\,\mathrm{d}\xi^3 \tag{38}$$

Equation (38) defines the two-dimensional moment tensor. Note that the vector of the moment acting on the plane of the section lies in a plane perpendicular to the preferred straight line.

KINEMATICS

The hypothesis that the straight line given by equation (29) remains straight after deformation implies that the deformation of the shell is uniquely determined by six displacement functions $u^a(\xi^1, \xi^2)$ and $r^a(\xi^1, \xi^2)$, whereby the post-deformation position vector $\hat{\tilde{x}}^a$ of any point within the shell is given as follows:

$$\hat{\tilde{x}}^a = f^a(\xi^1, \xi^2)+u^a(\xi^1, \xi^2)+\xi^3[g^a(\xi^1, \xi^2)+r^a(\xi^1, \xi^2)]. \tag{39}$$

The base vectors after deformation are:

$$\hat{\tilde{e}}^a_\alpha = \frac{\partial \hat{\tilde{x}}^a}{\partial \xi^\alpha} = \tilde{e}^a_\alpha + \frac{\partial u^a}{\partial \xi^\alpha} + \xi^3\frac{\partial r^a}{\partial \xi^\alpha} \tag{40}$$

and

$$\hat{\tilde{e}}^a_3 = \frac{\partial \hat{\tilde{x}}^a}{\partial \xi^3} = \tilde{e}^a_3 + r^a = e^a_3 + r^a. \tag{41}$$

The metric tensor after deformation, neglecting terms of order greater than 1, is given by the following three equations:

$$\hat{\tilde{G}}_{\alpha\beta} = \hat{\tilde{e}}^a_\alpha\hat{\tilde{e}}^a_\beta = \tilde{G}_{\alpha\beta} + e^a_\alpha\frac{\partial u^a}{\partial \xi^\beta} + e^a_\beta\frac{\partial u^a}{\partial \xi^\alpha} + \xi^3\left[\frac{\partial g^a}{\partial \xi^\alpha}\frac{\partial u^a}{\partial \xi^\beta} + \frac{\partial g^a}{\partial \xi^\beta}\frac{\partial u^a}{\partial \xi^\alpha} + e^a_\alpha\frac{\partial r^a}{\partial \xi^\beta} + e^a_\beta\frac{\partial r^a}{\partial \xi^\alpha}\right]$$

$$+(\xi^3)^2\left[\frac{\partial g^a}{\partial \xi^\alpha}\frac{\partial r^a}{\partial \xi^\beta} + \frac{\partial g^a}{\partial \xi^\beta}\frac{\partial r^a}{\partial \xi^\alpha}\right] \tag{42}$$

$$\hat{\tilde{G}}_{\alpha 3} = \hat{\tilde{e}}^a_\alpha\hat{\tilde{e}}^a_3 = \tilde{G}_{\alpha 3} + \tilde{e}^a_\alpha r^a + \frac{\partial u^a}{\partial \xi^\alpha}e^a_3 + \xi^3\frac{\partial r^a}{\partial \xi^\alpha}e^a_3 \tag{43}$$

$$\hat{\tilde{G}}_{33} = \hat{\tilde{e}}^a_3\hat{\tilde{e}}^a_3 = \tilde{G}_{33} + 2e^a_3 r^a. \tag{44}$$

The three-dimensional strain tensor is given by:

$$\tilde{E}_{ij} = \tfrac{1}{2}(\hat{\tilde{G}}_{ij} - \tilde{G}_{ij}). \tag{45}$$

Equations (42)–(45) yield the following expressions for the components of the strain tensor:

$$\tilde{E}_{\alpha\beta} = E_{\alpha\beta}+\tfrac{1}{2}\xi^3(K_{\alpha\beta}+K_{\beta\alpha})+\tfrac{1}{2}(\xi^3)^2\Big[B_{\alpha}^{\cdot\varphi}K_{\beta\varphi}+B_{\beta}^{\cdot\varphi}K_{\alpha\varphi}-2E_{\rho\varphi}B_{\alpha}^{\cdot\varphi}B_{\beta}^{\cdot\rho}$$

$$-2C_{\alpha}B_{\beta}^{\cdot\rho}E_{\rho 3}-2C_{\beta}B_{\alpha}^{\cdot\rho}E_{\rho 3}+C_{\alpha}\frac{\partial E_{33}}{\partial\xi^{\beta}}+C_{\beta}\frac{\partial E_{33}}{\partial\xi^{\alpha}}+2C_{\alpha}C_{\beta}E_{33}\Big] \tag{46}$$

$$\tilde{E}_{\alpha 3} = E_{\alpha 3}+\tfrac{1}{2}\xi^3\frac{\partial E_{33}}{\partial\xi^{\alpha}} \tag{47}$$

$$\tilde{E}_{33} = E_{33} \tag{48}$$

where

$$E_{\alpha\beta} = \frac{1}{2}\left(e_{\alpha}^{a}\frac{\partial u^{a}}{\partial\xi^{\beta}}+e_{\beta}^{a}\frac{\partial u^{a}}{\partial\xi^{\alpha}}\right) = E_{\beta\alpha} \tag{49}$$

$$K_{\alpha\beta} = \frac{\partial g^{a}}{\partial\xi^{\alpha}}\frac{\partial u^{a}}{\partial\xi^{\beta}}+e_{\beta}^{a}\frac{\partial r^{a}}{\partial\xi^{\alpha}} \tag{50}$$

$$E_{\alpha 3} = \frac{1}{2}\left(e_{\alpha}^{a}r^{a}+\frac{\partial u^{a}}{\partial\xi^{\alpha}}e_{3}^{a}\right) \tag{51}$$

$$E_{33} = e_{3}^{a}r^{a}. \tag{52}$$

$E_{\alpha\beta}$, $K_{\alpha\beta}$ and $E_{\alpha 3}$ can be viewed as generalized membrane and bending strain tensors and shear strain vector, respectively.

The rigidity requirement for the straight lines [equation (29)] is satisfied by setting:

$$E_{33} = 0. \tag{53}$$

Note that a less restrictive assumption is also possible.

The following kinematic unknowns will be adopted:

$$u_i = u^{a}e_{i}^{a} \tag{54}$$

$$r_i = r^{a}e_{i}^{a}. \tag{55}$$

Since, by equations (52) and (53)

$$r_3 = 0. \tag{56}$$

Equations (54) and (55) represent five independent unknowns.

INTERNAL ENERGY EXPRESSION

The internal virtual work is given exactly by:

$$IVW = \int_{\xi^1}\int_{\xi^2}\int_{\xi^3}\tilde{\sigma}^{ij}\delta\tilde{E}_{ij}\sqrt{(\tilde{G})}\,\mathrm{d}\xi^1\,\mathrm{d}\xi^2\,\mathrm{d}\xi^3 \tag{57}$$

or equivalently by:

$$IVW = \iint_A \left[\int_{\xi^3} (\tilde{\sigma}^{\alpha\beta}\delta\tilde{E}_{\alpha\beta} + 2\tilde{\sigma}^{\alpha 3}\delta\tilde{E}_{\alpha 3}) \sqrt{\left(\frac{\tilde{G}}{g}\right)}\, d\xi^3 \right] dA \tag{58}$$

where A = area of reference surface.

Using equations (34), (35), (38), (46), (47) and (53), equation (58) can be rewritten as:

$$IVW = \iint_A [\bar{N}^{\alpha\beta}\delta E_{\alpha\beta} + 2\bar{Q}^{\alpha}\delta E_{\alpha 3} + M^{\alpha\beta}\delta K_{\alpha\beta}]\, dA \tag{59}$$

where

$$\bar{N}^{\alpha\beta} = N^{\alpha\beta} - M^{\varphi\alpha}B_{\varphi}^{\cdot\beta} \tag{60}$$

$$\bar{Q}^{\alpha} = Q^{\alpha} - C_{\beta}M^{\beta\alpha}. \tag{61}$$

EQUILIBRIUM EQUATIONS

(a) *Force equilibrium*

Let the external load/unit area of the reference surface be given by:

$$p^a = p^i e_i^a. \tag{62}$$

The equilibrium of an arbitrary region of area A implies that:

$$\oint_C R^a\, ds + \iint_A p^a\, dA = 0 \tag{63}$$

where C is the boundary of the region under consideration.

Using equation (33) and surface divergence theorem, the following equation of force equilibrium in Cartesian components is obtained:

$$(N^{\beta\rho}e_{\rho}^a + Q^{\beta}e_3^a)_{;\beta} + p^a = 0. \tag{64}$$

Multiplying equation (64) by $e^{a\sigma}$ and by e^{a3}, the following equilibrium equations are obtained in the local three-dimensional base:

$$N^{\beta\sigma}_{;\beta} + N^{\beta\rho}A^{\sigma}_{\rho\beta} + Q^{\beta}B_{\beta}^{\cdot\sigma} + p^{\sigma} = 0 \tag{65}$$

and

$$N^{\beta\rho}D_{\rho\beta} + Q^{\beta}_{;\beta} + Q^{\beta}C_{\beta} + p^3 = 0. \tag{66}$$

(b) *Moment equilibrium*

Let the external moment per unit area of the reference surface be given by:

$$m^a = m_{\alpha}e^{a\alpha}. \tag{67}$$

The equilibrium of region A requires that:

$$\oint_C (\varepsilon^a_{.bc}x^bR^c + M^a)\, ds + \iint_A (\varepsilon^a_{.bc}x^bp^c + m^a)\, dA = 0. \tag{68}$$

Making use of equations (33), (37) and (64) and again applying the surface divergence theorem given by equation (27), the following moment equilibrium equation in Cartesian components is obtained:

$$\varepsilon^{a}_{.bc}x^{b}_{;\beta}(N^{\beta\rho}e^{c}_{\rho}+Q^{\beta}e^{c}_{3})+(\varepsilon_{\pi\omega 3}M^{\beta\pi}e^{a\omega})_{;\beta}+m^{a} = 0. \tag{69}$$

Multiplying equation (69) by e^{a}_{φ} and by e^{a}_{3}, the following moment equilibrium equations in the local three-dimensional base are obtained:

$$M^{\beta\pi}_{;\beta}+C_{\beta}M^{\beta\pi}+A^{\pi}_{\rho\beta}M^{\beta\rho}-Q^{\pi}+\varepsilon^{\pi\varphi 3}m_{\varphi} = 0 \tag{70}$$

and

$$\varepsilon_{\pi\rho 3}(N^{\pi\rho}-B^{\cdot\rho}_{\beta}M^{\beta\pi}) = 0. \tag{71}$$

Note that by equation (71)

$$\bar{N}^{\alpha\beta} = \bar{N}^{\beta\alpha} \tag{72}$$

where $\bar{N}^{\alpha\beta}$ is defined in equation (60).

(c) *Boundary conditions*

The normal boundary conditions are conveniently obtainable from the equality of the internal and external virtual works:

$$IVW = EVW \tag{73}$$

the former being given by equation (59), and the latter by:

$$EVW = \iint_{A}(p^{a}\delta u^{a}+m^{a}\delta\Omega^{a})\,\mathrm{d}A+\oint_{C}(R^{a*}\delta u^{a}+M^{a*}\delta\Omega^{*})\,\mathrm{d}s \tag{74}$$

where $\delta\Omega^{a}$ is the virtual rotation of e^{a}_{3}, in turn obtainable as:

$$\delta\Omega^{a} = \frac{\varepsilon^{a}_{.bc}e^{b}_{3}\delta r^{c}}{G_{33}} \tag{75}$$

and R^{a*} and M^{a*} are the force and moment resultants acting on the boundary.

It can be verified, by using equations (49)–(51), (59), (74) and (75), that equation (73) yields the same equilibrium equations as obtained earlier [equations (65), (66) and (70)] subject to the following normal boundary conditions:

$$\lambda_{\alpha}(N^{\alpha\beta *}-N^{\alpha\beta})\delta u_{\beta} = 0 \tag{76}$$

$$\lambda_{\alpha}(Q^{\alpha *}-Q^{\alpha})\delta u_{3} = 0 \tag{77}$$

$$\lambda_{\alpha}(M^{\alpha\beta *}-M^{\alpha\beta})\delta r_{\beta} = 0 \tag{78}$$

where $N^{\alpha\beta *}$, $Q^{\alpha *}$ and $M^{\alpha\beta *}$ are related to R^{a*} and M^{a*} by equations (33) and (37).

Equations (76)–(78) imply that either the forces and moments or the displacements (or linear combinations thereof) must be prescribed at the boundary.

It should be noted [8] that equation (71)—not obtained from the principle of virtual work—follows immediately from the definitions of $N^{\alpha\beta}$ in equation (34), and $M^{\alpha\beta}$ in equation (38) and from the symmetry of the stress tensor.

CONSTITUTIVE EQUATIONS

For linear elastic materials, the triaxial stress–strain relation is given by:

$$\tilde{\sigma}^{ij} = \tilde{T}^{ijkm}\tilde{E}_{km}. \tag{79}$$

Substituting equation (79) in equations (34), (35) and (38), and using equations (46)–(48) and (53) the following equations are obtained:

$$N^{\beta\alpha} = \int_{\xi^3}\sqrt{\left(\frac{\tilde{G}}{g}\right)}\{\tilde{T}^{\beta\varphi\gamma\rho}[E_{\gamma\rho}+\tfrac{1}{2}\xi^3(K_{\gamma\rho}+K_{\rho\gamma}+\tfrac{1}{2}(\xi^3)^2(B_{\gamma}^{\cdot\omega}K_{\rho\omega}+B_{\rho}^{\cdot\omega}K_{\gamma\omega}-2E_{\omega\lambda}B_{\gamma}^{\cdot\omega}B_{\rho}^{\cdot\lambda}$$
$$-2C_{\gamma}B_{\rho}^{\cdot\omega}E_{\omega 3}-2C_{\rho}B_{\gamma}^{\cdot\omega}E_{\omega 3})]+2\tilde{T}^{\beta\varphi\gamma 3}E_{\gamma 3}\}(\delta_{\varphi}^{\alpha}+\xi^3 B_{\varphi}^{\cdot\alpha})\,\mathrm{d}\xi^3 \tag{80}$$

$$Q^{\beta} = \int_{\xi^3}\sqrt{\left(\frac{\tilde{G}}{g}\right)}\{(\tilde{T}^{\beta 3\gamma\rho}+\xi^3 C_{\varphi}\tilde{T}^{\beta\varphi\gamma\rho})[E_{\gamma\rho}+\tfrac{1}{2}\xi^3(K_{\gamma\rho}+K_{\rho\gamma})$$
$$+\tfrac{1}{2}(\xi^3)^2(B_{\gamma}^{\cdot\omega}K_{\rho\omega}+B_{\rho}^{\cdot\omega}K_{\gamma\omega}-2E_{\omega\lambda}B_{\gamma}^{\cdot\omega}B_{\rho}^{\cdot\lambda}$$
$$-2C_{\gamma}B_{\rho}^{\cdot\omega}E_{\omega 3}-2C_{\rho}B_{\gamma}^{\cdot\omega}E_{\omega 3}]$$
$$+2(\tilde{T}^{\beta 3\gamma 3}+\xi^3 C_{\varphi}\tilde{T}^{\beta\varphi\gamma 3})E_{\gamma 3}\}\,\mathrm{d}\xi^3 \tag{81}$$

$$M^{\beta\alpha} = \int_{\xi^3}\sqrt{\left(\frac{\tilde{G}}{g}\right)}\{\tilde{T}^{\beta\varphi\gamma\rho}[E_{\gamma\rho}+\tfrac{1}{2}\xi^3(K_{\gamma\rho}+K_{\rho\gamma}$$
$$+\tfrac{1}{2}(\xi^3)^2(B_{\gamma}^{\cdot\omega}K_{\rho\omega}+B_{\rho}^{\cdot\omega}K_{\gamma\omega}-2E_{\omega\lambda}B_{\gamma}^{\cdot\omega}B_{\rho}^{\cdot\lambda}-2C_{\gamma}B_{\rho}^{\cdot\omega}E_{\omega 3}-2C_{\rho}B_{\gamma}^{\cdot\omega}E_{\omega 3})]$$
$$+2\tilde{T}^{\beta\varphi\gamma 3}E_{\gamma 3}\}(\delta_{\varphi}^{\alpha}+\xi^3 B_{\varphi}^{\cdot\alpha})\xi^3\,\mathrm{d}\xi^3. \tag{82}$$

Equations (80)–(82) relate the ten components of the biaxial forces and moments to the nine components of the two-dimensional strain tensors given by equations (49)–(51). Since equation (71) holds identically, however, there are only nine independent force and moment components.

Less exact constitutive equations are obtainable by postulating a quadratic form of the strain components for the internal energy expression. Equation (59) then provides directly the desired constitutive equations interrelating the nine stress components N^{11}, $\bar{N}^{12} = \bar{N}^{21}$, N^{22}, M^{11}, M^{12}, M^{21}, M^{22}, $\bar{Q}^1$ and $\bar{Q}^2$ and the nine strain components E_{11}, $E_{12} = E_{21}$, E_{22}, K_{11}, K_{12}, K_{21}, K_{22}, E_{13} and E_{23}.

SUMMARY

The following set of equations was obtained:

1. Kinematic relations [equations (49)–(51)] which can be rewritten in terms of the kinematic unknowns as follows:

$$E_{\alpha\beta} = \tfrac{1}{2}[u_{\alpha;\beta}+u_{\beta;\alpha}-2u_{\rho}A_{\alpha\beta}^{\rho}-2u_3 D_{\alpha\beta}] \tag{83}$$

$$E_{\alpha 3} = \tfrac{1}{2}[r_{\alpha}+u_{3;\alpha}-u_{\rho}B_{\alpha}^{\cdot\rho}-u_3 C_{\alpha}] \tag{84}$$

$$K_{\alpha\beta} = B_{\alpha}^{\cdot\omega}u_{\omega;\beta}-(B_{\alpha}^{\cdot\omega}A_{\beta\omega}^{\rho}+B_{\beta}^{\cdot\rho}C_{\alpha})u_{\rho}+C_{\alpha}u_{3;\beta}-(B_{\alpha}^{\cdot\omega}D_{\beta\omega}+C_{\alpha}C_{\beta})u_3+r_{\beta;\alpha}-r_{\rho}A_{\alpha\beta}^{\rho}. \tag{85}$$

2. Constitutive equations [equations (80)–(82)] which express the two-dimensional forces and moments as linear functions of the two-dimensional strains.

3. Equilibrium equations and boundary conditions [equations (65), (66), (70), (76)–(78)] that when expressed in terms of the kinematic unknowns by means of equations (80)–(85) constitute a system of five second order linear differential equations.

CONCLUSIONS

The theory presented has the following features:

1. It is based on the assumption that through every point of the reference surface a given material straight line, not necessarily normal to it, remains straight after deformation.
2. The shell is described in terms of the reference surface geometry and of several surface tensors representing the deviation of the straight line in question from the normal to the reference surface and the generalized curvature of it.
3. The deformation of the shell is uniquely determined by five independent kinematic unknowns.
4. The equilibrium equations are exact.
5. The boundary conditions permit the forces and moments at the boundary to be prescribed independently.
6. The equilibrium equations in terms of the kinematic unknowns are of the second order.

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Абстракт—Марчелом Эпстейном и Эйр Теном даются линейная теория тонких упругих оболочек основанная на законе сохранения не-нормальной прямой линии. Представляется полная линейая теория оболочек первого приближения, на основе кинематическбго предположения, что сквозь каждую точку плоскости отнесения, заданная материальная прямая линия, не конечно нормальная к этой плоскости, остается прямой после деформации. Даются уравнения в тензорной нотации.